Experiments on oscillator ensembles with global nonlinear coupling

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We experimentally analyze collective dynamics of a population of 20 electronic Wien-bridge limit-cycle oscillators with a nonlinear phase-shifting unit in the global feedback loop. With an increase in the coupling strength we first observe formation and then destruction of a synchronous cluster, so that the dependence of the order parameter on the coupling strength is not monotonic. After destruction of the cluster the ensemble remains nevertheless coherent, i.e., it exhibits an oscillatory collective mode (mean field). We show that the system is now in a self-organized quasiperiodic state, predicted in Rosenblum and Pikovsky [Phys. Rev. Lett. 98, 064101 (2007)]. In this state, frequencies of all oscillators are smaller than the frequency of the mean field, so that the oscillators are not locked to the mean field they create and their dynamics is quasiperiodic. Without a nonlinear phase-shifting unit, the system exhibits a standard Kuramoto-like transition to a fully synchronous state. We demonstrate a good correspondence between the experiment and previously developed theory. We also propose a simple measure which characterizes the macroscopic incoherence-coherence transition in a finite-size ensemble.

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For weak coupling and large $\gamma$, Eqs. (1) and (3) reduce to
\[ \phi_k = \omega + \bar{\epsilon} R \sin[\Theta - \phi_k + \beta(R, \bar{\epsilon})], \]
where $\bar{\epsilon} = \epsilon/\gamma$ and $\beta(R, \bar{\epsilon}) = \alpha + \eta^{-1} \bar{\epsilon}^2 R^2$ [8]. This phase equation differs from Eq. (2) by the dependence of the phase shift $\Phi_k$ on $R$, which is crucial for the dynamics. Indeed, let $|\alpha| < \pi/2$, then for small $\bar{\epsilon}$ the phase shift $\beta$ is also smaller than $\pi/2$, the system synchronizes, and $R = 1$. However, if the coupling increases beyond the critical value $\bar{\epsilon} > \epsilon_{cr} = \gamma(\pi/2 - \alpha)/\eta$, then $\beta$ becomes larger than $\pi/2$ and, hence, the interaction becomes repulsive. As a result, the system tends to desynchronize and to decrease the order parameter, which would make $\beta < \pi/2$, i.e., the interaction again would become attractive. Finally, the system settles exactly at the border between synchrony and asynchrony, with the order parameter $R < 1$ determined from the condition $\beta(R, \bar{\epsilon}) = \pi/2$. This desynchronization transition results in the divergence of frequencies of the mean field and of individual oscillators; generally these frequencies are incommensurate, and, hence, the dynamics of oscillators is quasiperiodic [8]. An analytical treatment of the model (4) was extended to the cases of Lorentzian [10] and uniform [11] distribution of frequencies. We briefly discuss the latter case, since it is closer to experimental implementation. With the increase of $\bar{\epsilon}$ first the transition to synchrony is observed. If the frequency distribution is sufficiently narrow then all oscillators form a synchronous cluster, otherwise part of them remains asynchronous. Next, oscillators, one by one, leave the synchronous cluster, and finally the SOQ state is formed, where all oscillators differ in frequency from the mean field they create. The transition from synchrony to SOQ is accompanied by a decrease in the order parameter. Although the theoretical treatment has been performed for phase oscillators, we expect that this effect can be observed for general limit-cycle oscillators, provided the phase shift in the global feedback monotonically depends on the mean-field amplitude.

![FIG. 1. Scheme of the experimental setup of $N = 20$ globally coupled oscillators. All oscillators have an identical structure and therefore only the first one is shown in detail. The global coupling is organized via the common load $R_c$. A fraction of the voltage across $R_c$ is fed back to each oscillator via the feedback loop, consisting of a linear (standard $RC$ circuit) and nonlinear phase-shifting units.](image1)

![FIG. 2. (Color online) Characteristic of the nonlinear phase-shifting unit: phase shift between the output and input vs the amplitude of the input.](image2)

![FIG. 3. (Color online) Results of the experiment with the linear phase-shifting unit. Order parameter $R$ (a) and minimal mean-field amplitude $A_{min}$ [(b), blue circles] reveal the synchronization transition at the coupling strength $\epsilon \approx 0.17$. This is also confirmed by the plot of $\eta$ [(b), red squares]: This quantity shows that for $\epsilon \gtrsim 0.17$, the instantaneous frequency of the mean field is always positive, as expected for a coherent, oscillatory mean field. The transition can be also very good seen from the frequency plot in (c): At $\epsilon \approx 0.17$ several oscillators form a synchronous cluster and for $\epsilon \gtrsim 0.2$ full frequency locking is observed, with $R$ close to 1. Here the circles show the frequencies of oscillators $f_i$ and the bold blue line shows the mean-field frequency $f_{mf}$. Notice that for subthreshold coupling $R$ is not small due to the finite-size effect; here $A_{mf}$ is more efficient for the determination of the threshold.](image3)
written as \( V_c = \varepsilon V_L \). The parameter \( \varepsilon \), \( 0 \leq \varepsilon \leq 1 \), quantifies the strength of the global coupling. Individual units are Wien-bridge oscillators with saturation of the amplitude due to the negative feedback loop of the operational amplifier (not shown) [12]. The input-output characteristic of the operational amplifier is a sigmoidal curve which can be approximated as \( V = ku \tanh(u/u_0) \), where \( u \) and \( V \) are the input and output voltages, \( u_0 = 3.2 \) V determines the range of the input voltages where the amplifier works without saturation, and \( k = 10 \) is the slope of the characteristics in the linear regime. All oscillators were tuned to have approximately the same output voltage \( \approx 1 \) V and close frequencies \( \approx 3.1 \) kHz. The phase-shifting unit has a linear and nonlinear part. The former is implemented via two standard \( RC \) circuits plus amplifiers, and the details of the latter are shown in Fig. 1; the characteristic of the nonlinear part is shown in Fig. 2.

We performed three experiments. In the first one the phase-shifting unit was excluded so that the signal from the common load was directly applied to the inputs of oscillators, i.e., \( V_I = V_c \). In the second experiment only the linear phase-shifting unit was included, and in the third run we had both linear and nonlinear units, as shown in Fig. 1. In each experiment we gradually changed the input to the feedback loop \( V_c \) from zero to its maximal value \( V_L \) and recorded the outputs of all oscillators, \( V_i \), \( i = 1, \ldots, N \), and the mean-field voltage \( V_m \) [13]. In each recording we obtained \( 10^5 \) points per channel, with the sampling rate 65 kHz. For each value of the coupling strength \( \varepsilon = V_c/V_L \) we performed ten recordings.

For the presentation of results we have computed, for each \( \varepsilon \), the following quantities: (i) Instantaneous phases \( \varphi_i \) of all oscillators and the instantaneous phase and amplitude \( A_{\text{inf}} \) of the mean field \( V_m \) were obtained with the help of the Hilbert transform; (ii) frequencies \( f_i \) of all oscillators and frequency \( f_{\text{inf}} \) of the mean field were computed from the unwrapped phases for each recording and then averaged over ten recordings; (iii) the order parameter \( \mathcal{R} \) was obtained by averaging the quantity \( N^{-1} \sum_{j=1}^{N} e^{i\phi_j} \) over time and over ten measurements; (iv) the minimal (over all ten measurements) value \( A_{\text{min}} \) of the instantaneous mean-field amplitude \( A_{\text{inf}} \); and (v) the fraction \( \eta \) of the data points where the instantaneous frequency of the mean field is negative. Typically, synchronization transition in a globally coupled system is traced by plotting \( \mathcal{R} \) vs \( \varepsilon \). In the limit \( N \to \infty \), \( \mathcal{R} = 0 \) in the incoherent state. However, since in our case \( N = 20 \), the finite-size fluctuations of the mean field in this state are quite large (they are known to scale as \( 1/\sqrt{N} \)) and therefore \( \mathcal{R} \) is not small either. We find that the distinction between incoherent (fluctuating mean field) and coherent (oscillatory mean field) states can be better revealed by \( A_{\text{min}} \) and \( \eta \) (see also the discussion of Fig. 5 below).

In the first and second experiments (no phase-shifting unit and linear unit, respectively), we observed standard Kuramoto transitions to collective synchrony. These transitions occurred at \( \varepsilon \approx 0.85 \) and \( \varepsilon \approx 0.17 \), respectively [14] (see Fig. 3, where the second experiment is illustrated), and were characterized by a monotonic dependence of \( \mathcal{R} \) and \( A_{\text{min}} \) on \( \varepsilon \). In the third, main experiment, we observed a nonmonotonic dependence of \( \mathcal{R} \) and \( A_{\text{min}} \) on \( \varepsilon \) (Fig. 4). We have found that with an increase of \( \varepsilon \), ten oscillators formed a cluster at \( \varepsilon \approx 0.12 \), while the other ten remained asynchronous. Next, the frequency-locked
oscillators left the cluster one by one. Finally, the SOQ state appeared at $\varepsilon \approx 0.72$. In order to show that this is indeed a transition to SOQ but not simply a breakup of synchrony, we plot in Figs. 5(a) and 5(b) the Hilbert transform of the mean field versus the mean field itself. We see that in the asynchronous state the pattern is typical for a narrowband random process, with the amplitude dropping practically to zero, whereas in the SOQ state the mean field is clearly oscillatory and its phase and frequency are well defined. The SOQ dynamics is illustrated by power spectra in Fig. 5(c).

Thus, we have experimentally demonstrated a state where oscillators are synchronized neither with each other nor with the mean field, but the amplitude of the latter is, nevertheless, nonzero. This peculiar coherent state is possible because of the combination of the mean field and its derivative, the zero phase shift is not optimal. Because of the combination of the mean field and its derivative, the zero phase shift is not optimal.

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[12] Components used are as follows: $R_1 = 270 \Omega$, $R_2 = 50 k \Omega$, $R_3 = 1 k \Omega$, $R_4 = 1.1 k \Omega$, $R_5 = 0.1 k \Omega$ (trimmer potentiometer was used to tune the oscillator frequency), $C_4 = 51 nF$, $C_3 = 100 nF$ (or time delay) on the synchronization threshold. In order to get insight as to why the phase shift in our setup enhances synchrony, we have derived the equations of the system. We obtained the van der Pol-type equations, driven by the common force $-\frac{v_i}{\epsilon} + V_i$. Because of the combination of the mean field and its derivative, the zero phase shift is not optimal.
