Reconstruction of time-delayed feedback systems from time series

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Abstract

For various classes of time-delay systems we propose the methods of their model delay-differential equation reconstruction from time series. The methods are based on the characteristic location of extrema in the time series of time-delay systems and the projection of infinite-dimensional phase space of these systems to suitably chosen low-dimensional subspaces. We verify our methods by using them for the recovery of time-delay differential equations from their chaotic solutions and for modelling experimental systems with delay-induced dynamics from their chaotic time series.

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1. Introduction

Systems, whose dynamics is affected not only by the current state, but also by past states, are wide spread in nature [1]. Usually these systems are modelled by delay-differential equations. Such models are successfully used in many scientific disciplines, like physics, physiology, biology, economics and cognitive sciences. Typical examples include population dynamics [2], where individuals participate in the reproduction of a species only after maturation, or spatially extended systems, where signals have to cover distances with finite velocities [3]. Within this rather broad class of systems, one can find the Ikeda equation [4] modelling the passive optical resonator system, the Lang–Kobayashi equations [5] describing semiconductor lasers with optical feedback, the Mackey–Glass equation [6] modelling the production of red blood cells, and many other
Generally, the time-delay systems are described by the following equation

\[ \varepsilon_n x^{(n)}(t) + \varepsilon_{n-1} x^{(n-1)}(t) + \cdots + \varepsilon_1 x(t) = F(x(t), x(t-\tau_1), \ldots, x(t-\tau_k)), \tag{1} \]

where \( x(t) \) is the system state at time \( t \), \( x^{(n)}(t) \) is the time derivative of order \( n \), \( \tau_1, \ldots, \tau_k \) are the delay times and \( \varepsilon_1, \ldots, \varepsilon_n \) are the parameters characterizing the inertial properties of the system. To uniquely define the system (1) behaviour it is necessary to prescribe the initial conditions in the entire time interval \([ -\tau_k, 0] \). Therefore, the phase space of the system has to be considered as infinite-dimensional. In fact, even first-order delay-differential equations can possess high-dimensional chaotic dynamics [8]. Thus, the direct reconstruction of the system by the time-delay embedding techniques runs into severe problems. For a successful recovery of the time-delay systems one has to use special methods. The most of them are based on the projection of the infinite-dimensional phase space of time-delay systems onto low-dimensional subspaces. These methods use different criteria of quality for the reconstructed equations, for example, the minimal forecast error of constructed model [9–12], the minimal value of information entropy [13] or various measures of complexity of the projected time series [14–18]. Several methods of time-delay system recovery exploit regression analysis [19–21].

In this paper we present the original procedure of the delay time reconstruction based on a statistical analysis of time intervals between extrema in the time series and develop further the methods of time-delay system parameter estimation from time series proposed by us recently [22,23] for a more wide class of time-delay systems including time-delay systems of high order and with several coexisting delays. We propose also the methods for reconstructing ring-type time-delay systems from time series of various dynamical variables obtained from different points of the time-delay system.

2. Peculiarities of time-delay system time series

Statistical analysis of time intervals between extrema in time series of various model and real time-delay systems reveals the following general regularities. If the system has inertial properties, the dependence of number \( N \) of pairs of extrema in its time series separated in time by \( \tau \) demonstrates a pronounced minimum at the level of the delay time of the system (Fig. 1(a)). Let us explain the qualitative features of \( N(\tau) \) with one of the most popular delay-differential equation

\[ \varepsilon_1 x(t) = -x(t) + f(x(t-\tau_1)). \tag{2} \]

In general case Eq. (2) is a mathematical model of an oscillating system composed of a ring with three ideal elements: nonlinear, delay, and inertial ones (Fig. 2).

In the presence of inertial properties \( \varepsilon_1 > 0 \), which corresponds to real situations, the extrema in \( x(t) \) are close to quadratic ones and therefore \( \dot{x}(t) = 0 \) and
followed within the time \( \tau \).

![Fig. 2](image)

Fig. 2. Block scheme of a ring system with nonlinear time-delayed feedback. Arabic numerals designate points where a dynamical variable can be measured.

\[ \dot{x}(t) \neq 0 \] at the extremal points. In fact, the condition \( \dot{x}(t) = \dot{x}(t) = 0 \) is satisfied for a point, which is a point of inflection, or a non-quadratic extremum, or belongs to an interval of constant value of the dynamical variable. But the presence of inertial properties in the system prevents the implementation of these conditions. It can be shown that in this case there are practically no extrema in \( x(t) \) separated in time by the delay time \( \tau \). Differentiation of Eq. (2) with respect to \( t \) gives

\[ \epsilon \dot{x}(t) = -\dot{x}(t) + \frac{\partial f(x(t - \tau_1))}{\partial t} \dot{x}(t - \tau_1). \]  

(3)

If for \( \dot{x}(t) = 0 \) in a typical case \( \dot{x}(t) \neq 0 \), then, as it can be seen from Eq. (3), for \( \epsilon \neq 0 \) the condition \( \dot{x}(t - \tau_1) \neq 0 \) must be fulfilled. Thus, there must be no extremum separated in time by \( \tau_1 \) from a quadratic extremum and, hence, \( N(\tau_1) \rightarrow 0 \).

Similar properties are inherent in a more general class of time-delay systems

\[ \dot{x}(t) = \dot{\bar{f}}(x(t), x(t - \tau_2)). \]  

(4)

Time differentiation of Eq. (4) gives

\[ \ddot{x}(t) = \frac{\partial \dot{\bar{f}}(x(t), x(t - \tau_2))}{\partial t} \dot{x}(t) + \frac{\partial \dot{\bar{f}}(x(t), x(t - \tau_2))}{\partial (t - \tau_1)} \dot{x}(t - \tau_1). \]  

(5)

Similarly to Eq. (3), Eq. (5) implies that in a typical case of quadratic extremum derivatives \( \dot{x}(t) \) and \( \dot{x}(t - \tau_1) \) do not vanish simultaneously, i.e., if \( \dot{x}(t) = 0 \), then \( \dot{x}(t - \tau_1) \neq 0 \).

In the absence of inertial properties \( (\epsilon_1 = 0) \) differentiation of Eq. (2) with respect to \( t \) gives

\[ \epsilon_2 \ddot{x}(t) = \frac{\partial f(x(t - \tau_1))}{\partial t} \dot{x}(t - \tau_1). \]  

(6)

From Eq. (6) it follows that if \( \dot{x}(t - \tau_1) = 0 \), then \( \dot{x}(t) = 0 \). Thus, for \( \epsilon_2 = 0 \) every extremum of \( x(t) \) is followed within the time \( \tau_1 \) by the extremum. As the result, \( N(\tau) \) shows a maximum for \( \tau = \tau_1 \) (Fig. 1(b)). If the system has a bounded bandpass \( (\epsilon_1 > 0) \), the most probable value of the time interval between extrema in \( x(t) \) shifts from \( \tau_1 \) to larger values and the extremum can be found most often at the distance \( \tau_1 + \tau_1 \) apart (Fig. 1(a)).

The presence of noise in time series brings into existence spurious extrema that are not caused by the intrinsic dynamics of a time-delay system. Thus, owing to high-frequency noise a probability to find a pair of extrema in time series separated in time by \( \tau \) has to increase in general. As a result, with noise increasing the average \( N(\tau) \) value becomes greater. The probability to find a pair of extrema separated by the interval \( \tau_1 \) also increases. However, for moderate noise levels this probability is still less than the probability to find a pair of extremum separated in time by \( \tau \neq \tau_1 \). Hence, the qualitative features of the \( N(\tau) \) plot specified by the delay-induced dynamics are retained for a moderate noise level.

### 3. Reconstruction of first-order time-delay systems

Let us consider the procedure of first-order time-delay system recovery with Eq. (2) as an example. To define the delay time \( \tau_1 \) one has to determine the extrema in the time series and after that to define for different values of the parameter \( \tau \) the number \( N(\tau) \) of pairs of extrema separated in time by \( \tau \) and to construct the \( N(\tau) \) plot.

The absolute minimum of \( N(\tau) \) located near the absolute maximum is observed at the delay time \( \tau_1 \). The dependence of accuracy of the delay time recovery on the step of \( \tau \) variation and the time series length is considered in Ref. [22].

To recover the parameter \( \tau_1 \) and the nonlinear function \( f \) from the chaotic time series let us rewrite Eq. (2) as

\[ \epsilon_2 \ddot{x}(t) + \dot{x}(t) = f(x(t - \tau_1)). \]  

(7)

Thus, it is possible to reconstruct the nonlinear function by plotting in a plane a set of points with coordinates \( (x(t - \tau_1), \dot{x}(t) + \dot{x}(t)) \). Since the parameter \( \epsilon_1 \) is a priori unknown, one needs to plot \( x(t) + x(t) \) versus \( x(t - \tau_1) \) under variation of \( \epsilon \), searching for a single-valued dependence in the \( (x(t - \tau_1), \dot{x}(t) + \dot{x}(t)) \) plane, which is possible only for \( \epsilon = \epsilon_0 \). As a quan-
titative criterion of single-valuedness in searching for $\varepsilon_1$ we use the minimal length of a line $L(\varepsilon)$ connecting all points ordered with respect to $x(t-\tau_1)$. The set of points constructed for the defined $\varepsilon_1$ in the plane $(x(t-\tau_1), \dot{x}(t) + x(t))$ reproduces the nonlinear function, which can be approximated if necessary. In contrast to methods presented in Refs. [15,16], which use only extremal points or points selected according to a certain rule for the nonlinear function recovery, the proposed technique uses all points of the time series. It allows one to estimate the parameter $\varepsilon_1$ and to reconstruct the nonlinear function from short time series even in the regimes of weakly developed chaos.

To test the efficiency of the proposed technique we have used it to reconstruct the equations of various time-delay systems having the form of Eq. (2) from the time series gained from their numerical solution. In particular, we apply the method to a time series of the Ikeda equation [4]

$$\dot{y}(t) = -y(t) + \mu \sin(y(t-\tau_1) - x_0)$$

modelling the passive optical resonator system. The Ikeda Eq. (8) has the form of Eq. (2) with $\varepsilon_1 = 1$. The parameters of the system (8) are chosen to be $\mu = 20$, $\tau_1 = 2$, $x_0 = \pi/3$ to produce a dynamics on a high-dimensional chaotic attractor [3]. Part of the time series is shown in Fig. 3(a). The time series is sampled in such a way that 200 points in time series cover a period of time equal to the delay time $\tau_1 = 2$. The data set consists of 20,000 points and exhibits about 1100 extrema.

For various $\tau$ values we count the number $N$ of situations when $\dot{x}(t)$ and $\dot{x}(t-\tau)$ are simultaneously equal to zero and construct the $N(\tau)$ plot (Fig. 3(b)). The step of $\tau$ variation in Fig. 3(b) is equal to the integration step $h = 0.01$. The time derivatives $\dot{x}(t)$ are estimated from the time series by applying a local parabolic approximation. The absolute minimum of $N(\tau)$ takes place exactly at $\tau = \tau_1 = 2.00$. To construct the $L(\varepsilon)$ plot (Fig. 3(c)) the step of $\varepsilon$ variation is also set by 0.01. The minimum of $L(\varepsilon)$ takes place accurately at $\varepsilon = \varepsilon_1 = 1.00$. In Fig. 3(d) the recovered nonlinear function is shown. It coincides practically with the true function of Eq. (8). Note, that for the construction of the $L(\varepsilon)$ plot and for the recovery of the multimodal function $f$ we use only 2000 points of the time series. For the approximation of the recovered function we use polynomials of different degree. The sinusoid amplitude (Fig. 3(d)) allows one to define the parameter $\mu$ of Eq. (8). The parameter $x_0$ can be calculated from the function value at $x(t-\tau_1) = 0$. The approximation of the recovered function with a polynomial of degree 20 allows us to obtain
the following estimation: \( \mu' = 19.94 \) and \( x_0' = 1.046 \) \((\pi/3 \approx 1.047)\).

To investigate the robustness of the method to perturbations we apply it to the data produced by adding a zero-mean Gaussian white noise to the time series of Eq. (8). For the case where the additive noise has a standard deviation of 20% of the standard deviation of the data without noise (the signal-to-noise ratio is about 14 dB) the location of the minimum of \( N(\tau) \) still allows us to estimate the delay time accurately, \( \tau_1' = 2.00 \). The minimum of \( L(\varepsilon) \) takes place at \( \varepsilon_1' = 0.98 \). The nonlinear function recovered using the estimated values \( \tau_1' \) and \( \varepsilon_1' \) is shown in Fig. 3(e). In spite of sufficiently high noise level and inaccuracy of estimation of \( \varepsilon_1 \) the recovery of the nonlinear function has a good quality, which is significantly higher than that reported in Ref. [21] for the same parameter values of the Ikeda equation with noise.

The second example is the method application to experimental time series of the electronic oscillator with delayed feedback. In the block representation of this oscillator (Fig. 2) a delay for time \( \tau_1 \) is provided by a delay line, the role of nonlinear element is played by an amplifier with the transfer function \( f \) and the system inertial properties are defined by a filter, which parameters specify \( \varepsilon_1 \). For the case when the filter is a low-frequency first-order \( RC \)-filter such oscillator is given by

\[
RCV(t) = -V(t) + f(V(t - \tau_1)), \quad (9)
\]

where \( V(t) \) and \( V(t - \tau_1) \) are the delay line input and output voltages, respectively; \( R \) and \( C \) are the resistance and capacitance of the filter elements. Eq. (9) is of form (2) with \( \varepsilon_1 = RC \).

At \( \tau_1 = 31.7 \) ms and \( \varepsilon_1 = 1.007 \) ms we record the signal \( V(t) \) (Fig. 4(a)) using an analog-to-digital converter with the sampling frequency \( f_s = 4 \) kHz. Since the delay time \( \tau_1 \) is not a multiple of the sampling time \( T_s = 0.25 \) ms, the recovery of \( \tau_1 \) cannot be absolutely accurate. For the step of \( \tau \) variation equal to \( T_s \), the absolute minimum of \( N(\tau) \) takes place at \( \tau_1' = 31.75 \) ms (Fig. 4(b)). The \( L(\varepsilon) \) plot, constructed with \( \tau_1' \) and the step of \( \varepsilon \) variation equal to 0.025 ms, demonstrates the minimum at \( \varepsilon_1' = 1.000 \) ms (Fig. 4(c)). The recovered nonlinear function (Fig. 4(d)) coincides practically with the true transfer function of the amplifier.

Besides the \( L(\varepsilon) \) plot we use two another criteria of quality for the system recovery. The first of them exploits synchronization of unidirectionally coupled time-delay systems. We try to synchronize the recovered model equation with the experimental system (9)
applying the following synchronization scheme [24]:

\[ \epsilon_1'(t) = -y(t) + f'(y(t - \tau'_1)) + k(V(t) - y(t)), \]  

(10)

where \( y(t) \) is the model variable, \( \tau'_1 \) and \( \epsilon'_1 \) are the recovered parameters, \( f' \) is the polynomial approximation of the reconstructed nonlinear function and \( k \) is the coupling coefficient. If the recovered model parameters are close to the true ones and the coupling coefficient \( k \) is sufficiently large, the response system (10) rapidly synchronizes with the driving system (9). To quantify the measure of synchronization we calculate the synchronization error \( \Delta = \langle |y(t) - y(t)| \rangle \), where \( \langle \cdot \rangle \) denotes a time average.

Fig. 5(a) shows the dependence of \( \Delta \) on the model parameter \( \tau'_1 \) varied in the vicinity of \( \tau'_1 = 31.75 \) ms with the step of variation equal to \( 0.5 \) ms. To construct this plot we use \( k = 0.5, \epsilon'_1 = 1.000 \) ms and approximation of the recovered function \( f' \) with a polynomial of degree 11. The value of \( \Delta \) is computed after transients and is averaged over 2.5 s. The minimum of \( \Delta \) is observed at \( \tau'_1 = 31.75 \) ms as well as the minimum of \( N(\tau) \). We also calculate the dependence of \( \Delta \) on \( \tau'_1 \) after adding Gaussian white noise to the time series of the driving system. For the noise level of 1% and 10% the minimum of \( \Delta(\tau'_1) \) still takes place at \( \tau'_1 = 31.75 \) ms (Fig. 5(a)). Note that for the construction of \( \Delta(\tau'_1) \) in the presence of noise we use \( \epsilon'_1 \) and \( f' \) recovered from noisy time series.

In a similar way we plot the dependence of \( \Delta \) on the model parameter \( \epsilon'_1 \) varied in the vicinity of \( \epsilon'_1 = 1.000 \) ms with the step of variation equal to 0.025 ms (Fig. 5(b)). The minimum of \( \Delta(\epsilon'_1) \) is observed at \( \epsilon'_1 = 0.975 \) ms which is slightly below the estimation of \( \epsilon'_1 \) obtained from the \( L(\epsilon) \) plot. This distinction between \( \epsilon'_1 \) estimations can result from inaccuracy of \( f' \) approximation performed for calculation of \( \Delta \). For the driving signal corrupted by additive Gaussian white noise of 1% and 10% the minimum of \( \Delta(\epsilon'_1) \) is again observed at \( \epsilon'_1 = 0.975 \) ms (Fig. 5(b)). As can be seen from Fig. 5, the inaccuracy of the delay time estimation gives a greater synchronization error than the inaccuracy of estimation of \( \epsilon_1 \).

The next quantitative measure of accuracy used in our paper for the recovered model is the one-step forecast error \( \sigma = \langle |y(t) - y(t)| \rangle \), where \( V(t) \) is the experimentally measured variable, \( y(t) \) is the variable of the model having the form of Eq. (10) with \( k = 0.000 \) and \( \langle \cdot \rangle \) denotes a time average. This measure shows how the model with the recovered \( \tau'_1, \epsilon'_1 \), and \( f' \) fits the observed data if the initial conditions for the one-step prediction are chosen from the experimental time series. The dependencies of \( \sigma \) on the parameters \( \tau'_1 \) and \( \epsilon'_1 \) are qualitatively similar to the dependencies \( \Delta(\tau'_1) \) and \( \Delta(\epsilon'_1) \) (Fig. 5), respectively, and are not shown here.

4. Peculiarities of ring time-delay system reconstruction

In the ring time-delay systems described by Eq. (2) a dynamical variable can be measured at different points indicated in Fig. 2 by the numerals 1–3. However, it should be mentioned that in the real systems it is not
always possible to localize the elements depicted in Fig. 2 or to choose the point of measurement because of the integrity of the system. The delayed feedback system recovery for the case when the observed dynamical variable is \( x(t) \) measured at the point 1 has been considered in Section 3.

In the case, when the observed dynamical variable is \( x(t - \tau_1) \) measured at the point 2 (Fig. 2), one can use the same procedure for estimation of the system parameters as in the case of \( x(t) \) measurement since the observable is simply shifted in time by the delay time \( \tau_1 \) about \( x(t) \). For example, reconstructing the electronic oscillator with delayed feedback described by Eq. (9) from experimental time series of voltage \( V(t - \tau_1) \) at the delay line output we obtain the results qualitatively similar to those presented in Fig. 4 for the case of the system recovery from the time series of \( V(t) \).

Let us consider a technique of the time-delay system (2) reconstruction for the third possible case, when the observed variable is \( f(x(t - \tau_1)) \) measured at the point 3 (Fig. 2). As well as in the time series of \( x(t) \), there are practically no extrema separated in time by \( \tau_1 \) in the time series of the variable \( f(x(t - \tau_1)) \), since \( f(x(t - \tau_1)) \) is a smooth function. Then, the delay time \( \tau_1 \) can be estimated by the location of the absolute minimum in the \( N(\tau) \) plot constructed from the variable \( f(x(t - \tau_1)) \).

The nonlinear function \( f \) can be recovered by plotting \( f(x(t - \tau_1)) \) versus \( x(t - \tau_1) \). To obtain the unknown values of \( x(t - \tau_1) \) one has to filter the chaotic time series of the variable \( f(x(t - \tau_1)) \) with a low-frequency first-order filter with the cut-off frequency \( \nu_0 = \frac{\nu}{t} \) and to shift the signal \( x(t) \) at the filter output by the delay time \( \tau_1 \) defined earlier. Since the parameter \( \epsilon_1 \) and correspondingly the value of \( \nu_0 \) are a priori unknown, we filter the time series of \( f(x(t - \tau_1)) \) versus \( x(t - \tau_1) \), where \( x(t - \tau_1) \) is the signal at the filter output shifted by the time \( \tau_1 \). Note, that a single-valued dependence in the plane \( (x(t - \tau_1), f(x(t - \tau_1))) \) is possible only for \( \epsilon = \epsilon_1 \). In this case \( x(t - \tau_1) = f(x(t - \tau_1)) \) and the set of points constructed in the plane reproduces the function \( f \), which can be approximated if necessary. As a quantitative criterion of single-valuedness in searching for \( \epsilon_1 \) we use again the minimal length of a line \( L(\epsilon) \) connecting all points in the plane \( (x(t - \tau_1), f(x(t - \tau_1))) \) ordered with respect to \( x(t - \tau_1) \). The minimum of \( L(\epsilon) \) is observed at \( \epsilon = \epsilon_1 \).

We apply the method to time series of the variable \( f(x(t - \tau_1)) \) of the Mackey–Glass equation [6]

\[
a(t) = -b a(t) + \frac{a(t - \tau_1)}{1 + x(t - \tau_1)}
\]

(11)

which can be converted to Eq. (2) by division by \( b \). The parameters of the system (11) are chosen to be \( a = 0.2 \), \( b = 0.1 \), \( c = 10 \) and \( \tau_1 = 300 \) to produce a dynamics on a high-dimensional chaotic attractor [8]. Part of the time series is shown in Fig. 6(a). The location of the absolute minimum of \( N(\tau) \) (Fig. 6(b)) allows us to recover the delay time; \( \tau_1 = 300 \). The step of \( \tau \) variation in Fig. 6(b) is equal to the integration step \( b = 1 \). The minimum of \( L(\epsilon) \) (Fig. 6(c)) takes place at \( \epsilon_1 = 10.0 (\epsilon_2 = 1/b = 10) \). To construct the \( L(\epsilon) \) plot we use the step of \( \epsilon \) variation equal to 0.1. The nonlinear function recovered using the estimated \( \epsilon_1 \) and \( \epsilon_2 \) (Fig. 6(d)) coincides practically with the true nonlinear function.

5. Reconstruction of time-delay systems of high order

The method of the delay time definition based on the statistical analysis of time intervals between extrema in the time series can be extended to time-delay systems of high order

\[
e_{\alpha} x^{(\alpha)}(t) + e_{\alpha-1} x^{(\alpha-1)}(t) + \cdots + e_1 x(t) = F(x(t), x(t - \tau_1)).
\]

(12)

Time differentiation of Eq. (12) gives

\[
e_{\alpha} x^{(\alpha+1)}(t) + e_{\alpha-1} x^{(\alpha+1)}(t) + \cdots + e_1 x'(t) = \frac{\partial F(x(t), x(t - \tau_1))}{\partial x(t)} x'(t) + \frac{\partial F(x(t), x(t - \tau_1))}{\partial x(t - \tau_1)} x'(t - \tau_1).
\]

(13)

For \( x(t) = 0 \) the condition \( x(t - \tau_1) \neq 0 \) will be satisfied if the left-hand side of Eq. (13) does not vanish. If a probability to obtain zero in the left-hand side of Eq. (13) is very small for the extremal points, the \( N(\tau) \) plot qualitatively must have a shape similar to that inherent in the case of first-order delay-differential equations such as Eqs. (2) and (4).

We have found out that for sufficiently small values of \( e_i, i = 1, \ldots, n \), the \( N(\tau) \) plot demonstrates the absolute minimum at \( \tau = \tau_1 \) as well in the case of the
first-order time-delay systems. The distribution of the values of the left-hand side of Eq. (13) at the extremal points has a pronounced minimum in the neighborhood of zero in this case. As the parameters \(\varepsilon\) increase, the absolute minimum of \(N(\tau)\) shifts from \(\tau_1\) to larger values. The greater are \(\varepsilon\) characterizing the influence of the system inertia elements, the greater is the shift. This time shift of \(N(\tau)\) minimum does not depend on \(\tau_1\). Note that in the first-order time-delay systems the location of the absolute minimum in the \(N(\tau)\) plot does not depend on \(\varepsilon\).

The proposed method of the parameter \(\varepsilon_1\) estimation and the nonlinear function recovery based on the projection of infinite-dimensional phase space of the time-delay system to suitably chosen two-dimensional subspaces can be also applied to a variety of time-delay systems of order higher than that of Eq. (2). For instance, if the dynamics of a time-delay system is governed by the second-order delay-differential equation

\[
e_2 x(t) + e_1 x(t) = -x(t) + f(x(t - \tau_1)),
\]

(14)

the nonlinear function can be reconstructed by plotting in a plane a set of points with coordinates \((x(t), e_2 x(t) + e_1 x(t) + x(t))\). Since the parameters \(e_1\) and \(e_2\) are a priori unknown, one needs to plot \(\hat{e}_1 x(t) + \hat{e}_2 x(t) + x(t)\) versus \(x(t)\) under variation of \(\hat{e}_1\) and \(\hat{e}_2\), searching for a single-valued dependence, which is possible only for \(\hat{e}_1 = e_1, \hat{e}_2 = e_2\). This search for \(e_1\) and \(e_2\) we calculate the length of a line \(L(\hat{e}_1, \hat{e}_2)\) connecting points ordered with respect to \(x(t)\) in the plane \((x(t), \hat{e}_1 x(t) + \hat{e}_2 x(t) + x(t))\). The minimum of \(L(\hat{e}_1, \hat{e}_2)\) is observed at \(\hat{e}_1 = e_1, \hat{e}_2 = e_2\). The set of points constructed in the plane for these defined values of \(e_1\) and \(e_2\) reproduces the nonlinear function.

The methods of reconstruction of second-order time-delay systems from scalar time series have been considered in Refs. [10,17,18]. However, these methods deal only with the recovery of the delay time and the nonlinear function. For the recovery of the latter one they use only the points of the phase space section. As the result, these methods need long time series for qualitative reconstruction of the nonlinear function.

The proposed by us procedure of the delay time estimation based on the statistical analysis of time intervals between extrema in the time series needs significantly smaller time of computation than the methods of the delay time definition based on calculation of the fill-
Fig. 7. Reconstruction of the electronic oscillator with delayed feedback with a two-section filter. (a) Number \( N \) of pairs of extrema in the experimental time series separated in time by \( \tau \) normalized to the total number of extrema. \( N_{\text{min}}(\tau) = N(31.75 \text{ ms}) \). (b) The \( L(\hat{\epsilon}_1, \hat{\epsilon}_2) \) plot normalized to the number of points. \( L_{\text{min}}(\hat{\epsilon}_1, \hat{\epsilon}_2) = L(1.48 \text{ ms}, 0.48 \text{ ms}^2) \). (c) The recovered nonlinear function.

To verify the method efficiency we have applied it to experimental time series gained from the electronic oscillator with delayed feedback that is similar to that considered in Section 3, but contains two identical low-frequency in-series RC-filters. The dynamics of this oscillator is governed by Eq. (14), where \( x(t) \) and \( x(t - \tau_1) \) are the delay line input and output voltages, respectively, \( \epsilon_1 = R_1 C_1 + R_2 C_2 \), \( \epsilon_2 = R_1 C_1 R_2 C_2 \), and \( R_1, R_2, C_1 \) and \( C_2 \) are respectively the resistances and capacitances of the first and the second filters.

Using the analog-to-digital converter we record with the sampling frequency 4 kHz the time series of voltage at the delay line input for \( \tau_1 = 31.7 \text{ ms}, R_1 C_1 = 1.007 \text{ ms} \) and \( R_2 C_2 = 0.479 \text{ ms} \) (\( \epsilon_1 = 1.486 \text{ ms} \) and \( \epsilon_2 = 0.482 \text{ ms}^2 \)). The absolute minimum of \( N(\tau) \) is observed at \( \tau_{\text{min}} = 31.75 \text{ ms} \) (Fig. 7(a)). The \( L(\hat{\epsilon}_1, \hat{\epsilon}_2) \) plot, constructed with the step of \( \hat{\epsilon}_1 \) variation equal to 0.01 ms and the step of \( \hat{\epsilon}_2 \) variation equal to 0.01 ms\(^2\), demonstrates the minimum at \( \hat{\epsilon}_1 = 1.48 \text{ ms} \) and \( \hat{\epsilon}_2 = 0.48 \text{ ms}^2 \) (Fig. 7(b)). These \( \hat{\epsilon}_1 \) and \( \hat{\epsilon}_2 \) values give the following estimation of the filter parameters: \( (R_1 C_1) = 1.00 \text{ ms} \) and \( (R_2 C_2) = 0.48 \text{ ms} \). The recovered nonlinear function (Fig. 7(c)) coincides with a good accuracy with the true transfer function of the nonlinear element.

With the recovered parameters \( \hat{\epsilon}_1 \) and \( \hat{\epsilon}_2 \) we plot the distribution of the sum \( \hat{\epsilon}_1^2 x(t) + \hat{\epsilon}_2 \ddot{x}(t) \) using all points of the time series (Fig. 8(a)). The maximum of this distribution is observed close to zero. The distribution of the same sum constructed using only the extremal points \( \dot{x}(t) = 0 \) demonstrates a pronounced minimum in the vicinity of zero (Fig. 8(b)). This result counts in favour of the conclusion that the probability to obtain zero in the left-hand side of Eq. (13) is sufficiently small at the extremal points. The presence of minimum in the vicinity of zero in Fig. 8(b) agrees well with the existence of minimum at the delay time of the system in the \( N(\tau) \) plot (Fig. 7(a)).

Fig. 8. Distribution \( D \) of the sum \( \hat{\epsilon}_1 x(t) + \hat{\epsilon}_2 \ddot{x}(t) \) for all points of the experimental time series (a) and for the extremal points (b).
The next example is the method application to time series of the third-order time-delay system
\[ \varepsilon_3 \dot{x}(t) + \varepsilon_2 \dot{x}(t) + \varepsilon_1 x(t) = -x(t) + f(x(t - \tau_1)) \]  
(15)
with quadratic nonlinear function \( f(x) = \lambda - x^2 \), where \( \lambda \) is the parameter of nonlinearity. The parameters of the system (15) are chosen to be: \( \tau_1 = 300 \), \( \lambda = 1.9 \), \( \varepsilon_1 = 4\), \( \varepsilon_2 = 5 \) and \( \varepsilon_3 = 2 \). The \( N(\tau) \) plot, constructed with the step of \( \tau \) variation equal to unity, shows the absolute minimum at \( \tau_1^* \approx 301 \) (Fig. 9a). The minimum of \( N(\tau) \) tends to \( \tau_1 = 300 \) as the parameters \( \varepsilon_i \) decrease and shifts to larger \( \tau \) as \( \varepsilon_i \) increase. For example, we obtain \( \tau_1^* \approx 300 \) at \( \varepsilon_1 = 2.5 \), \( \varepsilon_2 = 2 \), \( \varepsilon_3 = 0.5 \), and \( \tau_1^* \approx 302 \) at \( \varepsilon_1 = 8 \), \( \varepsilon_2 = 17 \) and \( \varepsilon_3 = 10 \). The higher is the order of the time-delay system (12), the more parameters are to be fitted. This problem is typical in high-dimensional search space [25]. As the result, the time of computation significantly increases. Since our procedure of the parameters estimation involves numerical calculation of the derivatives, the quality of reconstruction deteriorates with the increase of the time-delay system order, resulting in the necessity to calculate more high-order derivatives. In Fig. 9b the recovered nonlinear function of Eq. (15) is shown. The quality of this function recovery is worse than the quality of reconstruction for the first-order time-delay systems (8) and (11) (Figs. 3 and 6).

6. Recovery of time-delay systems with two coexisting delays

Let us consider now a time-delay system with two different delay times \( \tau_1 \) and \( \tau_2 \)
\[ \dot{x}(t) = F(x(t), x(t - \tau_1), x(t - \tau_2)). \]  
(16)
Differentiation of Eq. (16) with respect to \( \tau \) gives
\[ \dot{x}(t) = \left. \frac{\partial F}{\partial t} \right|_{\tau = \tau_1} \dot{x}(t) + \left. \frac{\partial F}{\partial (t - \tau_1)} \right|_{\tau = \tau_1} \dot{x}(t - \tau_1) + \left. \frac{\partial F}{\partial (t - \tau_2)} \right|_{\tau = \tau_1} \dot{x}(t - \tau_2). \]  
(17)
Similarly to temporal realization of Eq. (4), the realization \( x(t) \) of Eq. (16) has mainly quadratic extrema and therefore \( \dot{x}(t) = 0 \) and \( \ddot{x}(t) \neq 0 \) at the extremal points. Hence, if \( \ddot{x}(t) = 0 \), the condition
\[ a(t - \tau_1) + b(t - \tau_2) \neq 0 \]  
(18)
must be fulfilled, where \( a = \left. \frac{\partial F}{\partial t} \right|_{\tau = \tau_1} \), \( \dot{x}(t - \tau_1) \), \( x(t - \tau_2) \), \( \dot{x}(t - \tau_1) \), \( \dot{x}(t - \tau_2) \) and \( b = \left. \frac{\partial F}{\partial (t - \tau_1)} \right|_{\tau = \tau_1} \), \( x(t - \tau_2) \), \( \dot{x}(t - \tau_2) \). The condition (18) can be satisfied only if \( \dot{x}(t - \tau_1) \neq 0 \) or \( \ddot{x}(t - \tau_2) \neq 0 \). By this is meant that the derivatives \( \dot{x}(t) \) and \( \ddot{x}(t - \tau_1) \), or \( \ddot{x}(t) \) and \( \ddot{x}(t - \tau_2) \) do not vanish simultaneously. As the result, the number of extrema separated in time by \( \tau_1 \) and \( \tau_2 \) from a quadratic extremum must be appreciably less than the number of extrema separated in time by other values of \( \tau \) and hence, the \( N(\tau) \) plot will demonstrate pronounced minima at \( \tau = \tau_1 \) and \( \tau = \tau_2 \). Compared to the method of optimal transformations used in Ref. [19] for the recovery of two delays our method requires longer time series, but it is significantly more simple and does not need preprocessing of the data as for example, adaptive partitioning of data used in Ref. [19].

We illustrate the procedure for estimating the other characteristics of time-delay system with two delays from time series for the system governed by the following equation
\[ \varepsilon_1 \dot{x}(t) = -x(t) + f_1(\dot{x}(t - \tau_1)) + f_2(\dot{x}(t - \tau_2)). \]  
(19)
Differentiation of Eq. (19) with respect to \( t \) gives

\[
e_1 x(t) = \frac{d}{dt}(x(t - t_1)) + \frac{\partial f(x(t - t_1))}{\partial x} (x(t - t_1)) \]

From Eq. (20) it follows that if

\[
x(t - t_1) = x(t - t_2) = 0
\]

then \( e_1 x(t) = -\dot{x}(t) \) and

\[
e_1 = -\frac{\dot{x}(t)}{x(t)}.
\]  

Thus, to estimate the parameter \( e_1 \) one can find the points of \( x(t) \) satisfying condition (21), define for them the first and the second derivatives, calculate \( e_1 \) using Eq. (22), and conduct averaging. Note that one can also use Eq. (22) for the recovery of \( e_1 \) in the case of a single delay, but such estimation uses only the points with \( x(t - t_1) = 0 \) and is not so accurate as the method considered in Section 3. To reduce the computation time we use Eq. (22) for the first approximation of the parameter \( e_1 \) and improve this estimation later on.

To recover the nonlinear functions \( f_1 \) and \( f_2 \) we project the trajectory generated by Eq. (19) to a three-dimensional space \( x(t - t_1), x(t - t_2), e_1 x(t) + x(t) \). In this space the projected trajectory is confined to a two-dimensional surface since according to Eq. (19)

\[
e_1 x(t) + x(t) = f_1(x(t - t_1)) + f_2(x(t - t_2)).
\]  

The points of this surface with the \( x(t - t_2) = \text{const} \) plane enable one to recover the nonlinear function \( f_1 \) up to a constant since the points of the section are correlated via \( e_1 x(t) + x(t) = f_1(x(t - t_1)) + e_1 \), where \( e_1 = f_2(x(t - t_2)) \) for some fixed value of \( x(t - t_1) \). In a similar way one can recover up to a constant the nonlinear function \( f_2 \) by intersecting the trajectory projected to the above-mentioned three-dimensional space with the \( (t - t_1) = \text{const} \) plane. The points of this section are correlated via \( e_1 x(t) + x(t) = f_2(x(t - t_2)) + e_2 \), where \( e_2 = f_1(x(t - t_1)) \) for fixed \( x(t - t_2) \).

We demonstrate the method efficiency with a generalized Mackey–Glass equation obtained by introducing a further delay,

\[
x(t) = -\dot{x}(t) + \frac{1}{T} \sum_{k=1}^{N} a_k x(t - \tau_k).
\]  

Division of Eq. (24) by \( b \) reduces it to Eq. (19) with \( e_1 = 1/b \). Fig. 10(a) shows the \( A(t) \) plot for \( a_1 = 0.2, a_2 = 0.3, b = 0.1, c = 10, \tau_1 = 70 \) and \( \tau_2 = 300 \). The first two most pronounced minima of \( A(t) \) are observed at \( \tau'_1 = 69 \) and \( \tau'_2 = 300 \). Another distinctive minimum of \( A(t) \) is observed close to \( \tau = \tau_1 + \tau_2 \). Processing the points satisfying condition (21) with the recovered values \( \tau'_1 \) and \( \tau'_2 \) we obtain the averaged estimation \( e_1' = 9.4 \) for the parameter \( e_1 = 1/b = 10 \). To reduce inaccuracy in \( e_1 \) determination by formula (22) we exclude from consideration the points with very small values of \( x(t) \).

To achieve more high quality of the model equation reconstruction we propose the following procedure for the correction of the parameters. Varying \( \tau_1 \) in a small vicinity of \( \tau'_1 = 69 \) we project the time series to several three-dimensional \( (x(t - \tau_1), x(t - \tau_2), e_1 x(t) + x(t)) \) and constructing the sections of this space with the planes \( x(t - \tau_2) = \text{const} \) and \( x(t - \tau_1) = \text{const} \) we obtain at these sections the recovered nonlinear functions \( f_1 \) and \( f_2 \) (Fig. 10(b) and (c)). However, as the result of inaccuracy in estimation of \( \tau_1 \) and \( e_1 \) the quality of the nonlinear function recovery is not good enough.

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Fig. 10. (a) Number $N$ of pairs of extrema in the time series of Eq. (24) separated in time by $\tau$. $N(\tau)$ is normalized to the total number of extrema in the time series. (b) The recovered nonlinear function $f_1$. (c) The recovered nonlinear function $f_2$.

Fig. 11. (a) Length $L$ of a line connecting points ordered with respect to abscissa in the $x(t - \tau)^2 = 1$ section, as a function of $\tau$. $L(\tau) = L(70)$. (b) Length $L$ of a line connecting points ordered with respect to abscissa in the $x(t - \hat{\tau}_1)^2 = 1$ section, as a function of $\hat{\tau}_1$. $L(\hat{\tau}_1) = L(300)$. (c) Length $L$ of a line connecting points ordered with respect to abscissa in the $x(t - \hat{\tau}_2)^2 = 1$ section, as a function of $\epsilon$. $L(\epsilon) = L(10.1)$. (d) Nonlinear function $f_1$ recovered up to the constant $\hat{c}_1 = f_2(x(t - \hat{\tau}_2)^2)$, where $x(t - \hat{\tau}_2)^2 = 1$. (e) Nonlinear function $f_2$ recovered up to the constant $\hat{c}_2 = f_1(x(t - \hat{\tau}_1)^2)$, where $x(t - \hat{\tau}_1)^2 = 1$. 


the embedding spaces with the $x(t - \hat{\tau}_2) = \text{const}$ plane with the corrected delay time $\hat{\tau}_2$. Successive correction
of $\tau_1$ and $\tau_2$ is continued until the parameters cease changing. For small deviations of initial estimates $\tau_1'$ and $\tau_2'$ from the true delay times the procedure is converging and allows one to define both delay times accurately.

After revision of the delay times the parameter $\varepsilon_1$ should be corrected. Its new estimate $\hat{\varepsilon}_1$ can be obtained by formula (22). However, a more reliable estimation is the one using all points of one of the section. To obtain it we project the time series to $(x(t - \hat{\tau}_1), x(t - \hat{\tau}_2), \dot{x}(t) + x(t))$ spaces under variation of $\varepsilon$ in the vicinity of $\varepsilon_1'$, searching for a single-valued dependence in the section $x(t - \hat{\tau}_1) = \text{const}$ or in the section $x(t - \hat{\tau}_2) = \text{const}$. The $L(\varepsilon)$ plot shows the minimum at $\hat{\varepsilon}_1 = 10.1$ (Fig. 11(c)). In Fig. 11 the values of $L(\varepsilon_1, \tau_1)$ and $L(\tau_2)$ are normalized to the number of points in the corresponding section. Note, that the proposed procedure of the successive correction of the parameters needs in several orders of magnitude smaller time of computation than the method of simultaneous selection of the parameters $\varepsilon_1$, $\tau_1$ and $\tau_2$ for the three-dimensional embedding space $(x(t - \tau_1), x(t - \tau_2), \dot{x}(t) + x(t))$.

As another example, we consider the method application to experimental time series produced by a setup with two delays. A block diagram of the electronic scheme is shown in Fig. 12. This electronic oscillator is governed by Eq. (23), where $x(t)$ is the voltage at the input of the delay lines, $x(t - \tau_1)$ and $x(t - \tau_2)$...
are the output voltages of the first and the second delay lines, respectively, and \( \epsilon_1 = RC \). The time series of \( V(t) \) are recorded at \( \tau_1 = 23.0 \) ms, \( \tau_2 = 31.1 \) ms and \( \epsilon_1 = 1.007 \) ms with the sampling frequency \( f_s = 4 \) kHz. The \( N(t) \) plot, constructed with the step of \( \tau \) variation equal to the sampling time \( T_s = 0.25 \) ms, demonstrates the first two most pronounced minima at \( \tau_1' = 23.0 \) ms and \( \tau_2' = 31.0 \) ms (Fig. 13(a)). These values of the delay times allow us to obtain the estimation \( \epsilon_1' = 1.16 \) ms from Eqs. (21) and (22). To obtain the estimation of \( \epsilon_1 \) using more number of points we project the time series to \( (V(t - \tau_1'), V(t - \tau_2'), V(t) + V'(t)) \) spaces under variation of \( \epsilon \) in the vicinity of \( \epsilon_n \), searching for a single-valued dependence in the section \( V(t - \tau)' \) = const. The \( Li(\epsilon) \) plot, constructed with the step of \( \epsilon \) variation equal to 0.01 ms, demonstrates the minimum at \( \tilde{\epsilon}_1 = 1.01 \) ms (Fig. 13(b)). The recovered nonlinear functions \( f_1 \) and \( f_2 \) are presented in Fig. 13(c) and (d), respectively, for \( \tau_1' = 23.0 \) ms, \( \tau_2' = 31.0 \) ms and \( \tilde{\epsilon}_1 = 1.01 \) ms. These functions are recovered up to a constant and are sufficiently close to the true transfer functions of the nonlinear elements of the scheme.

7. Conclusion

We have proposed the methods for reconstructing various classes of time-delay systems from chaotic time series. These methods are based on the statistical analysis of time intervals between extrema in the time series and the projection of infinite-dimensional phase space of the time-delay system to suitably chosen low-dimensional subspaces. The methods can be applied to the systems of different nature if these systems have similar structure of model equations. The proposed techniques allow one to estimate the delay times, the parameters characterizing the inertial properties of the system and the nonlinear functions even in the presence of sufficiently high noise. The method of delay time definition uses only operations of comparison and adding. It needs neither ordering of data, nor calculation of approximation error or certain measure of complexity of the trajectory and therefore it does not need significant time of computation. For the systems with a single delay time the procedures proposed for the nonlinear function recovery and estimation of the parameters characterizing the inertial properties of the system use all points of the time series. It allows one to successively apply the method to short time series even in the regimes of weakly developed chaos.

It is shown that the model equations of the ring time-delay systems can be reconstructed from time series of various dynamical variables measured at different points of the time-delay system. The methods efficiency is illustrated by the reconstruction of time-delay differential equations from their time series including the case of noise presence and by modelling real time-delayed feedback systems from experimental data.

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